The Cone of Semisimple Monoids with the same Factorial Hull

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Abstract

The *factorial hull* of the projective variety X (or its cone) is a graded algebra $R(X)$ that can be used in some situations to consider simultaneously all divisor classes on X. In this paper we consider initially the situation where X is a *semisimple variety* associated with the semisimple monoid M. The factorial hull of such X is determined by a certain arrangement $\mathbb{H}$ of hyperplanes in the space of rational characters $X(T)\otimes\mathbb{Q}$ of a maximal torus T of G_0. If G_0 is simply connected $R(X)$ is the coordinate ring of Vinberg's enveloping monoid $Env(G_0)$. Associated with X is a certain cone $\mathcal{H} \subseteq Cl(X)$ in the class group of X. Each $\delta \in \mathcal{H}$ corrresponds to a semisimple monoid M_δ with $R(M_\delta) = R(X)$. M and N have the same factorial hull if X_M and X_N differ by $G \times G$-orbits of codimension two or more. We calculate $\mathcal{H}$ explicitly in the case where X is the *plongement magnifique* for the simple goup G_0. This is exactly the case where M is a *canonical monoid*.

1 The $\mathbb{Q}$-Factorial Hull

The $\mathbb{Q}$-factorial hull of a projective variety is easy to describe, assuming it exists. This construction is related to a well-known question of Hilbert [11, 13, 18, 31]. Assume that X is an irreducible, normal, projective variety over the algebraically closed field K of characteristic zero. Let $Cl(X)$ be the divisor class group of X, and assume that $F \subseteq Cl(X)$ is a free abelian subgroup of finite rank. Choose representatives M_α, $\alpha \in F$, consisting of rank-one, locally reflexive sheaves on X. By results of [4, 1, 5], there is a natural graded K-algebra structure on

$$R_F(X) = \bigoplus_{\alpha\in\Gamma} R(X)_\alpha$$

where $R(X)_\alpha = \Gamma(X, M_\alpha)$ is the space of global sections of M_α.

In the case where $F = Cl(X)$ (so that $Cl(X)$ is free and finitely generated) we write $R(X)$ for $R_F(X)$. The interesting issue here is to find useful conditions on X and F which guarantee that $R_F(X)$ is finitely generated as a K-algebra. In case $Cl(X)$ is finitely generated but not free, there is often an "obvious" free subgroup $F \subseteq Cl(X)$ of finite index that one would like to consider in the discussion.

Definition 1.1. We say that the projective variety X has a *factorial hull* if $Cl(X)$ is finitely generated and free, and $R(X)$ is finitely generated as a K-algebra. If $Cl(X)$ if merely finitely generated and there is a free abelian subgroup $F \subseteq Cl(X)$ of finite index such that $R_F(X)$ is a finitely generated K-algebra, we refer to $R_F(X)$ as a $\mathbb{Q}$-*factorial hull* of X.

In this paper it will often suffice to think in terms of a $\mathbb{Q}$-factorial hull, since our main result is the calculation of a certain rational poyhedral cone $\mathcal{H}$ in the class group of X. See Theorems 4.2 and 4.3.

$\Gamma(X, M_\alpha)$ can often be described directly in terms of divisors and functions on X. If X has a factorial hull then X can be described as the categorical quotient of an open subset U of $Spec(R(X))$ by an algebraic torus. Furthermore, by the results of [5], $R(X)$ is a unique factorization domain, hence our terminology *factorial hull*. Other terminologies here are **Cox ring** [4], and **total homogeneous coordinate ring** [5]. Hausen [7] has shown that, in some cases, this factorial hull can be used to provide greater flexibility in geometric invariant theory by allowing certain Weil divisors instead of restricting only to ample Cartier divisors. $\mathbb{Q}$-factorial projective varieties with factorial hull are characterized in [8] using Mori theory. It is known that any spherical variety has a $\mathbb{Q}$-factorial hull. However, nobody seems to have published any proof. Brion has described a construction that depicts each $\mathbb{Q}$-factorial hull as a fibre product over the appropriate versal object, similar to how Vinberg [30] obtains all flat, reductive monoids from $Env(G_0)$. Indeed, $Env(G_0)$ is a $\mathbb{Q}$-factorial hull of the **plongement magnifique** [3]. Alternately, one can use a result of Knop [10] to obtain the $\mathbb{Q}$-factorial hull of a spherical variety by observing that, for X spherical, $R_F(X)$ is the ring of global functions on some closely related spherical variety.

Our cone $\mathcal{H}$ appears to be what is called the **moving cone** in [8]. Unfortunately it is beyond the scope of this paper to discuss in detail the very interesting relationship between our results and those of [8]. In fact our main results are independent of the discussion of factorial hulls or Mori theory. We mention those results mainly to place our work in a more general context that could be appealing to some readers. Clearly there is potential for further interesing work along these lines.

In some cases, (e.g. toric varieties [4], reductive monoids [24], the E_6 cubic surface [6]) $R(X)$ can be constructed geometrically from X by realizing $Spec(R(X))$ as the total space of a type of universal "$\mathbb{G}_m^r$-torsor" over X. In the case of reductive monoids a factorial hull had been already constructed indirectly in [22]. See Theorem 1.2 below.

Now let M be a **semisimple** monoid [24] with unit group G. By definition, M is reductive and normal, M has a zero element $0 \in M$, and $\dim Z(G) = 1$. Let $X = X_M = (M \backslash \{0\})/K^*$ be the associated **semisimple variety** [23]. See also [29].

We now describe the $\mathbb{Q}$-factorial hull of $X = X_M$. It suffices to descibe the $\mathbb{Q}$-factorial hull of M (whose coordinate algebra is a "sufficiently good" summand of $R(X)$ so that M and $X(M)$ have the same $\mathbb{Q}$-factorial hull). For the details of this construction we refer to Theorem 6.7 of [24]. We let $\Lambda^1(M)$ be the set of codimension-one $G \times G$-orbits of M.

Theorem 1.2. *Let M be reductive with unit group G. Then there exists a reductive monoid $\widehat{M}$ and a morphism $\pi : \widehat{M} \to M$ such that*

(i) $Cl(\widehat{M}) = (0)$,

(ii) π induces a bijection of $\Lambda^1(\widehat{M}) \to \Lambda^1(M)$.

Furthermore, the unit group $\widehat{G}$ of $\widehat{M}$ is an extension of G by a D-group.

Remark 1.3. $\widehat{M}$ was first constructed in [22] before the author noticed that such monoids are necessarily see **flat** (§ 2). The main problem originally motivating the present paper is the following question.

Let M be a semisimple monoid with unit group G. How does one characterize/classify, up to isomorphism, the set of semisimple monoids N such that $\widehat{N} = \widehat{M}$? What finer structure does this set $\mathcal{H}$ of monoids have?

Although these questions might seem far removed from anything geometric, the final answer is most naturally phrased in terms of the divisor class group of M or $X_M = (M\backslash\{0\})/K^*$, since this class group can be identified with a group of characters on the center $\widehat{M}$. See Theorems 3.4, 4.2 and 4.3. Theorem 3.4 discusses the case of a general semisimple monoid, while Theorems 4.2 and 4.3 discuss the case of the monoids (canonical monoids) associated with the plongement magnifique.

Remark 1.4. The construction of $\widehat{M}$ involves a (noncanonical) finite, dominant morphism $\pi : G' \to G$ with $Cl(G') = (0)$. Since the latter is not canonical, neither is $\widehat{M}$. To obtain a canonical object one needs to work a little deeper, and also be satisfied with a $\mathbb{Q}$-factorial hull.

Remark 1.5. Let M be a semisimple monoid with associated semisimple variety $X_M = (M\backslash\{0\})/K^*$. Let $\Lambda^1 = \{[D_i]\} \subseteq Cl(X_M)$ be the set of divisor classes of codimension-one $G \times G$-orbits of X_M. Let $V \subseteq Cl(X_M)$ be the subgroup generated by Λ^1. Then V is a torsion-free subgroup of $Cl(X)$ of finite index. Furthermore, $Cl(M) = V \oplus Cl(G)$. $R_V(X_M) = R_V(M)$ is a $\mathbb{Q}$-factorial hull of X_M. Furthermore $K[M] \subseteq R_V(X_M)$ and $M_V = Spec(R_V(X_M))$ is a reductive normal algebraic monoid whose unit group is an extension of the unit group of M by an algebraic torus. See §2 below for another construction, called $M_{\mathbb{H}}$, in terms of the hyperplanes in $X(T_0)$ determined by $\{[D_i]\}$. It turns out that there is a canonical isomorphism $M_V \to M_{\mathbb{H}}$.

2 Flat Monoids and Hyperplane Arrangements

Let M be a semisimple monoid. In this section we discuss the construction $M \rightsquigarrow M_{\mathbb{H}}$ (and related constructions) from several points of view. On the one hand, it turns out that $M_{\mathbb{H}}$ is determined by a certain rational, oriented, hyperplane arrangement in $X(T_0)$ (a **W-arrangment**). On the other hand, any reductive monoid with trivial divisor class group is **flat** in the sense of Vinberg [30]. We determine how the W-arrangement $\mathbb{H}$ of M essentially determines the monoid $M_{\mathbb{H}}$. Indeed, the theory of flat monoids allows us to construct $M_{\mathbb{H}}$ directly from $\mathbb{H}$.

2.1 Flat Reductive Monoids

Associated with any reductive monoid M, is its **abelization**

$$\pi : M \to A.$$

In [30] Vinberg calls M **flat** if π is a flat morphism with reduced and irreducible fibres. An important observation here (see Theorem 2.1 below) is that any reductive monoid M with trivial divisor class group is flat.

Let G be the unit group of M, and let G_0 be the semisimple part of G. Let B and B^- be opposite Borel subgroups of G containing the maximal torus T with unipotent radicals B_u and B_u^- respectively. Let $T_0 = T \cap G$ be the associated maximal torus of G_0.

Let $X(T_0)_+$ denote the monoid of dominant weights of T_0. If $\lambda \in X(T_0)_+$, we can write

$$\lambda = \sum_{\alpha \in \Delta} c_\alpha \lambda_\alpha,$$

where $\{\lambda_\alpha\}$ is the set of fundamental dominant weights of G_0. Define

$$c : X(T_0)_+ \to Cl(M)$$

by $c(\lambda) = \sum_{\alpha \in \Delta} c_\alpha [\overline{Bs_\alpha B^-}]$. Here $s_\alpha \in S$ is the simple involution corresponding to $\alpha \in \Delta$. Let

$$L(M) = \{ f \in K[M] \mid f(ugv) = f(g) \text{ for all } u \in B_u,\ v \in B_u^- \text{ and } f(1) = 1 \}.$$

Let $Z \subseteq G$ be the connected center of G so that $G = ZG_0$ and let $\overline{Z} \subseteq M$ be the Zariski closure of Z in M. $X(\overline{Z})$ is the set of characters of $\overline{Z}$.

Theorem 2.1. *Let M be a reductive monoid with unit group G, and let G_0 be the semisimple part of G. Assume that M has a zero element. The following are equivalent.*

a) The abelization morphism $\pi : M \to A$ is flat, with reduced and irreducible fibres.

b) The following two conditions hold.

i) If $\chi_1\lambda_1 = \chi_2\lambda_2$ ($\lambda_i \in \mathcal{M}$, $\chi_i \in X(\overline{Z})$) then $\chi_1 = \chi_2$ and $\lambda_1 = \lambda_2$.

ii) $\mathcal{M}$ is a subsemigroup of $L(M)$.

c) The canonical map $c : X(T_0)_+ \to Cl(M)$ is trivial.

d) For any irreducible representation $\rho : M \to End(V)$ there is a character $\chi : \overline{Z} \to K$ of $\overline{Z}$, and an irreducible representation $\sigma : M \to End(V)$, such that $\sigma(e) \neq 0$ for any $e \in \Lambda^1$ and $\rho = \chi \otimes \sigma$.

e) Any $f \in L(M)$ factors as $f = \chi g$ where $\chi \in X(\overline{Z})$, and $g \in Ł(M)$ has zero set $Z(g) \subseteq \cup_{\alpha \in \Delta} \overline{Bs_\alpha B^-}$.

For the proof see Theorem 6.10 of [24]. Notice in particular, if the divisor class group of M is trivial, that M is flat (using part c) of Theorem 2.1 above).

It turns out that that there is a universal, flat monoid $Env(G_0)$ associated with each semisimple group G_0. This amazing monoid was originally discovered and constructed by Vinberg in [30]. He refers to it as the **enveloping semigroup** of G_0. It has the following universal property.

Let M be any flat monoid with zero. Assume that the semisimple part of the unit group of M is G_0. Let $A(M)$ denote the abelization of M, and let $\pi_M : M \to A(M)$ be the abelization morphism. We make one exception with this notation. We let A denote the abelization of $Env(G_0)$ and we let $\pi : Env(G_0) \to A$ be the abelization morphism. The universal property of $Env(G_0)$ is as follows. Given any isomorphism φ_0 from the semisimple part of $G(M)$ to the semisimple part of $Env(G_0)$, there are unique morphisms

$$a : A(M) \to A$$

and

$$\varphi : M \to Env(G_0)$$

such that

i) $\varphi|G_0 = \varphi_0$;

ii) $a \circ \pi_M = \pi \circ \varphi$;

iii) $\phi : M \cong E(a, \pi)$, via $\phi(x) = (\pi_M(x), \varphi(x))$, where $E(a, \pi) = \{(x, y) \in A(M) \times Env(G_0) \mid a(x) = \pi(y)\}$, is the **fibred product** of $A(M)$ and $Env(G_0)$ over A.

There are several ways to construct this monoid $Env(G_0)$, and there are already hints in Theorem 2.1. However, we use the construction in Theorem 17 of Rittatore's thesis [25]. The reader should also see Vinberg's construction in [30]. A similar construction, due to Rittatore [26], exists in positive characteristics. Notice that we are using multiplicative notation for characters. In particular, $X(A) \cong P$ is the submonoid of $X(T_0)$ generated by the positive roots.

Assume that M has unit group G. Since $G \subseteq M$ is open we obtain that

$$K[M] \subseteq K[G].$$

Now, it is well known that

$$K[G] = \bigoplus_{\lambda \in X_+} K[G]_\lambda$$

where X_+ is the set of dominant characters of T with respect to B. Here, each $K[G]_\lambda$ is an irreducible $G \times G$-module with highest weight $\lambda \otimes \overline{\lambda}$ and $G \times G$ acts on G via $((g, h), x) \mapsto gxh^{-1}$. This "multiplicity ≤ 1" condition implies that any $G \times G$-stable subspace of $K[G]$ is a sum of some of the $K[G]_\lambda$. In particular,

$$K[M] = \bigoplus_{\lambda \in L(M)} K[G]_\lambda,$$

where $L(M) \subseteq X_+$. We refer to $L(M)$ as the **augmented cone** of M. It is defined, in a different but equivalent way, in Section 2.1.

Theorem 2.2. *Let G_0 be a semisimple group and let*

$$\mathfrak{L}(G_0) = \{(\chi, \lambda) \in L(T_0 \times G_0) \mid \chi\lambda^{-1} \in P\}.$$

Define

$$K[Env(G_0)] = \bigoplus_{(\chi,\lambda)\in\mathfrak{L}(G_0)} (V_\lambda \otimes V_\lambda^*) \otimes \chi \subseteq K[G_0 \times T_0].$$

Then $K[Env(G_0)]$ is the coordinate algebra of the normal, reductive algebraic monoid $Env(G_0)$ with the above-mentioned universal property. In particular, $L(Env(G_0)) = \mathfrak{L}(G_0)$.

For the proof see Theorem 6.16 of [24].

If G_0 is simply connected then $K[Env(G_0)]$ is the factorial hull of any **canonical monoid** of G_0 [16].

Closer scrutiny of this universal property yields a numerical classification of the flat monoid M in terms of a certain map $\theta_M^* : X(T_0)_+ \to X(\overline{Z})$. The above-mentioned classifying map $a : A(M) \to A$ can then be calculated directly.

We view $X(T_0)$ as a multiplicative group and write the simple roots exponentially $\{e^\alpha \mid \alpha \in \Delta\} \subseteq X(T_0)$. Recall that $X(A) \cong P$, the free commutative submonoid of $X(T_0/Z_0)$ generated by the positive roots. Thus the coordinate ring of A is the polynomial ring with the universal generators $\{u_\alpha \cong e^\alpha \mid \alpha \in \Delta\}$.

Corollary 2.3. *Let M be flat with unit group $G = G_0 Z$ and connected center $Z \subseteq G$. Let $Z_0 = Z \cap G_0$.*

1. *M is determined by a certain map $\theta_M^* : X(T_0)_+ \to X(\overline{Z})$ such that*

   (a) $\theta | Z_0 = id$,

   (b) θ_M^* extends to $\theta_M^* : X(T_0) \to X(Z)$ with $\theta_M^*(e^\alpha) \in X(A_M) = X(\overline{Z}/Z_0) \subseteq X(\overline{Z})$ for all $\alpha \in \Delta$.

   Conversely, any $\theta^* : X(T_0)_+ \to X(\overline{Z})$ satisfying a) and b) above determines a flat monoid $M = M_\theta$ with unit group G.

2. *The augmented cone of M_θ is determined by θ as follows.*

$$L(M_\theta) = \{(\chi, \lambda) \in L(Z \times G_0) \mid \chi\theta_M^*(\lambda)^{-1} \in X(A)\}.$$

3. *To obtain M as a fibred product from $\pi : Env(G_0) \to A$ define*

$$a : A_M \to A$$

   by the rule

$$a^*(u_\alpha) = \theta_M^*(e^\alpha).$$

   Then, as above, $\phi : M_\theta \cong E(a, \pi)$ via $\phi(x) = (\pi_M(x), \varphi(x))$.

The above corollary is a reformulation of Theorems 4 and 5 of [30]. See also the proof of Theorem 6.16 in [24]. Recall from Theorem 2.2 that

$$L(Env(G_0)) = \mathcal{L}(G_0) = \{(\chi, \lambda) \in L(T_0 \times G_0) \mid \chi\lambda^{-1} \in P\}.$$

In this case, $X(\overline{Z}) = \{\lambda \in X(T_0) \mid \lambda^n \in P \text{ for some } n > 0\}$ and $\theta^* : X(T_0)_+ \to X(Z)$ is just the inclusion.

There is another useful characterization of flat monoids. Let M be a reductive monoid and let $\overline{T} \subseteq M$ be the closure in M of a maximal torus of G. Define

$$X(\overline{T})_+ = \{\chi \in X(\overline{T}) \mid \Delta_\alpha(\chi) \geq 0 \text{ for all } \alpha \in \Delta\},$$

where Δ_α is defined by the equation

$$\Delta_\alpha(\chi)\alpha = \chi - \sigma_\alpha(\chi).$$

For each $e \in \Lambda^1$ there is a "valuation" $\nu_e : X(T) \to \mathbb{Z}$ determined by the divisor $e\overline{T} \subseteq \overline{T}$ (ν_e is induced by the inclusion $K^* \subseteq T$ of the 1-PSG containing e in its closure).

Define

$$\mathcal{M} = \{\lambda \in X(\overline{T})_+ \mid \nu_e(\lambda) = 0 \text{ for all } e \in \Lambda^1\}.$$

Theorem 2.4. *Let*

$$r : \mathcal{M} \to X(T_0)_+$$

be defined by $r(\lambda) = \lambda|T_0$. The following are equivalent.

1. *$r : \mathcal{M} \to X(T_0)_+$ is an isomorphism.*
2. *M is flat*

Furthermore, in this case, $\theta^* = p \circ r^{-1}$, where $p : \mathcal{M} \to X(\overline{Z})$ is defined by $p(\lambda) = \lambda|\overline{Z}$.

Proof. Assume that M satisfies condition *1* above. If $\lambda \in X(T)_+$ then there is a unique $\lambda_0 \in \mathcal{M}$ such that $r(\lambda) = r(\lambda_0)$. So we let $\delta = \lambda\lambda_0^{-1}$. It follows easily that $\delta \in X(A) \subseteq X(\overline{T})$. This gives us the desired factorization $\lambda = \delta\lambda_0$ as in part e) of Theorem 2.1. $\square$

2.2 The Flat Monoid $M_{\mathbb{H}}$ of an Arrangement

Let M be semisimple monoid with unit group G and let T be a maximal torus of G. Recall that $E(\overline{T}) = \{e \in \overline{T} \mid e^2 = e\}$. $E(\overline{T})$ is a poset if we define $e \geq f$ whenever $ef = f$. Let

$$E^1 = \{e \in E(\overline{T}) \mid e \text{ is maximal in } E(\overline{T})\backslash\{1\}\}.$$

E^1 is refered to as the set of **maximal idempotents** of $\overline{T}$. E^1 corresponds bijectively to the set of codimension-one $T \times T$-orbits $\{eT\}$ of $\overline{T}$. Associated with each $eT \subseteq \overline{T}$ there is the associated valuation v_e of $K[T]$. Let $Z \subseteq G$ be the connected center of G. In the following definition we let ν_e be the restriction of v_e to $X(T/Z) \subseteq X(T)$. ν_e is dual to the 1-psg $\lambda : K^* \to T/Z$ that has $\lim\limits_{t\to 0}\lambda_t \in (eT)/Z$.

Definition 2.5. Let M be as above.

a) The *arrangement* $\mathbb{H}(M)$ of M is the collection

$$\mathbb{H}(M) = \{\nu_e \mid e \in E^1\}.$$

b) Let

$$\Lambda^1\mathbb{H}(M) = \{\nu \in \mathbb{H}(M) \mid \nu(-\alpha) \geq 0 \text{ for all } \alpha \in \Delta\}.$$

$\Lambda^1\mathbb{H}(M)$ is a fundamental domain for the action of the Weyl group on $\mathbb{H}(M)$. $\Lambda^1\mathbb{H}(M)$ can be identified with $\Lambda^1 = \{e \in E^1(\overline{T}) \mid Be = eBe\ \}$.

We identify $\mathbb{H}(M)$ and $\mathbb{H}(N)$ if there is a bijection $f : \mathbb{H}(M) \to \mathbb{H}(N)$ such that, if $f(l) = m$, then $ker(l) = ker(m)$ and l is a positive multiple of m. Notice also that if $l, m \in \mathbb{H}(M)$ and $ker(l) = ker(m)$, and l is a *positive* multiple of m, then $l = m$. This is a natural nondegeneracy condition inherent in a semisimple monoid. We conclude that $\mathbb{H}(M)$ can be thought of as a collection of rational, oriented, W-invariant hyperplanes $\{(H, l)\}$ in $X(T/Z) \otimes \mathbb{Q}$.

By abuse of language we often write $\mathbb{H}(M) = \{(ker(\nu_e), \nu_e) \mid e \in E^1\}$, if we wish to emphasize the rôle of $ker(\nu_e)$. In any case, $\mathbb{H}(M)$ can be thought of as either a set oriented hyperplanes in $X(T/Z)$, or as a set of rays in $Hom_{\mathbb{Z}}(X(T/Z), \mathbb{Z})$.

If $w \in W$ the action on $\mathbb{H}(M)$ is determined by

$$w(H, l) = (w(H), l \circ w^{-1}).$$

Notice also that we may identify $X(T/Z) \otimes \mathbb{Q}$ with $X(T_0) \otimes \mathbb{Q}$, where $T_0 \cap G_0 \subseteq G_0$ is the associated maximal torus of the the semisimple part of G.

We now construct $M_{\mathbb{H}}$ from an arrangement $\mathbb{H}$ using Corollary 2.3. Let $\Lambda^1\mathbb{H}$ be a finite set of hyperplanes in $X = X(T_0)$ such that for each $H \in \Lambda^1\mathbb{H}$, there is a functional $l : X \to \mathbb{Z}$ such that

1. $l(-\alpha) \geq 0$ for all $\alpha \in \Delta$,

2. $ker(l) = H$.

Notice that $Span_{\mathbb{Z}}(\Delta) = X(T/Z_0) \subseteq X$ is a subgroup of finite index. Notice also that l is determined up to a positive scalar by H. So each H is canonically oriented. We then define $\mathbb{H}$ as follows.

$$\mathbb{H} = \{(w(H), l \circ w^{-1}) \mid w \in W\}.$$

Now for each $H \in \Lambda^1\mathbb{H}$, there is a unique ("best") functional $l : X \to \mathbb{Z}$, as above, which is not an integer multiple of any other. So we can write (H, l) for H. Write $\Lambda^1\mathbb{H} = \{(H_1, l_1), (H_2, l_2), ..., (H_s, l_s)\}$.

Let $\pi : Env(G_0) \to A$ be the abelization morphism of $Env(G_0)$. Notice that the set of characters $X(A)$ of A is canonically identified with the submonoid $< \Delta >$ of $X(A^*)$.

Thus, $< \Delta >= P = X(A) \subseteq X(T_0/Z_0) = X(A^*) = Span_{\mathbb{Z}}(\Delta) \subseteq X(T_0) = X$. Hence each functional $l : X \to \mathbb{Z}$ restricts to a functional $l : X(A) \to \mathbb{Z}$. Define

$$a^* : X(A) \to \mathbb{N}^s$$

by $a^*(\chi) = (-l_1(\chi), -l_2(\chi), ..., -l_s(\chi))$. This makes sense, since for each $\alpha \in \Delta$ and each i, $l_i(-\alpha) \geq 0$. Define

$$X(A_{\mathbb{H}}) = \oplus_{i=1}^{s} k_i \mathbb{N} \subseteq \mathbb{N}^s$$

where each integer $k_i > 0$ is chosen to be maximal subject to the condition that $a^*(X(A)) \subseteq X(A_{\mathbb{H}})$.

Thus we obtain, by definition, that

$$a : A_{\mathbb{H}} \to A.$$

Definition 2.6. We let $M_{\mathbb{H}} = E(a, \pi)$ be the fibre product of a over π (as in Corollary 2.3). $M_{\mathbb{H}}$ is called the *flat monoid of the arrangement* $\mathbb{H}$.

Remark 2.7. The purpose of this remark is to indicate in more detail how θ^* and a^* determine each other. Suppose that M is flat. Then for all $e \in E^1(A)$ there is a unique $\chi \in X(A_M)$ such that $eA_M = \chi^{-1}(0)$. Furthermore, this defines a "valuation" $\nu_e : L(G) \to \mathbb{Z}$ since $\pi^{-1}(eA_M)$ is a codimension-one $G \times G$-orbit of M. Now

$$\mathcal{M} = \{(\theta^*(\lambda), \lambda) \mid \lambda \in X(T_0)_+\}.$$

Thus,

$$\nu_e((1, \lambda)) = \nu_e((\theta^*(\lambda)^{-1}, 1)(\theta^*(\lambda), \lambda)) = \nu_e(\theta^*(\lambda)^{-1}, 1)$$

since, by Theorem 2.4, $\nu_e(g) = 0$ for all $g \in \mathcal{M}$ and $e \in E^1(A_M)$. Observe also that

$$\nu_e((1, e^{-\alpha})) = \nu(\theta^*(e^{\alpha}), 1) \geq 0$$

since $(\theta^*(e^{\alpha}), 1) \in X(A_M) \subseteq X(\overline{Z})$.

If $M \cong E(a, \pi)$, as in Definition 2.6, one can check that $a^* : P \to X(A_M)$ extends uniquely to a map $b^* : X(Z(Env(G_0))) \to X(\overline{Z})$. Furthermore,

$$L(M) = \{(\delta b^*(\lambda), \lambda) \mid \lambda \in X(T_0)_+, \delta \in X(A_M)\}.$$

The canonical map $M \cong E(a, \pi) \to Env(G_0)$ induces the map $\gamma : L(Env(G_0)) \to L(M)$ defined by $\gamma(\chi, \lambda) = (b^*(\chi), \lambda)$.

Recall from Example 1.5 the definition of M_V.

Theorem 2.8. *There is a canonical finite, dominant morphism $M_V \to M_{\mathbb{H}}$.*

Proof. The canonical restriction map $Cl(M_V) \to Cl(G)$ is an isomorphism, by construction of M_V, since $ker(Cl(M_V) \to Cl(G))$ is the subgroup generated by $\{[D_i]\}$ as in Remark 1.5. However, by construction of M_V, each $[D_i] \in Cl(M_V)$ is zero. Also the composite $X(T_0)_+ \underset{c}{\to}$

$Cl(M) \to Cl(G)$ is the trivial homomorphism for any reductive, normal monoid. Thus $c : X(T_0)_+ \to Cl(M_V)$ is also trivial. So M_V is flat by Theorem 2.1. Thus by Corollary 2.3 there is a unique classifying map

$$h^* : X(A) \to X(A_{M_V})$$

such that $M_V \cong E(h,\pi)$. The assumption that $Cl(M_V) = Cl(G)$ is equivalent to the statement that for each $e \in \Lambda^1$ the ideal $\{f \in K[M_V] \mid f \text{ vanishes on } GeG\}$ is a principal ideal (χ_e) of $K[M]$. Hence $X(A_M) = \mathbb{N}^s$ is the free abelian monoid on these (normalized) generators. Let $p_e : X(A_M) \to \mathbb{N}$ be the projection onto the i-th factor and let $h_e^* = p_e \circ h^*$. Let $X \subseteq X(A_{M_V})$ be the smallest submonoid of $X(A_M)$ of the form

$$X = \oplus_{e \in \Lambda^1} k_e \mathbb{N}$$

so that $h^*(X(A)) \subseteq X$. Since M_V is a monoid with $|\Lambda^1| = |\mathbb{H}|$ one can check that $h^* : X(A) \to X$ is isomorphic to $a^* : X(A) \to X(A_{\mathbb{H}})$, and that $K[X] \subseteq K[X(A_M)]$ is a finite morphism. This says that the classiying map for M factors through the classifying map for $M_{\mathbb{H}}$. Thus there is a canonical finite morphism $M \cong E(h,\pi) \to M_{\mathbb{H}}$. Furthermore $E(h,\pi) \cong E(a, \pi_{\mathbb{H}})$. $\square$

Corollary 2.9. *Let M be a reductive, normal monoid with unit group G and zero element $0 \in M$. Suppose that $Cl(M) = Cl(G)$. Then M is flat.*

3 The Coterie of an Arrangement

In this section we shall often work with rational cones and vector spaces. But we want our notation to be consistent with all previous notation. If L is some lattice of interest we denote by L_0 the rational vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$. If $P \subseteq L$ is a finitely generated, additive submonoid of L then P_0 is the rational cone in L_0 generated by P (In this paper P_0 is the rational polyhedral cone generated by the positive roots). We make one exception with this notation. Script quantities (esp. $\mathcal{H}$, $\mathcal{C}$ and $\mathcal{C}(\delta)$; see below) automatically represent rational entities.

The purpose of this section is to determine a useful criterion for identifying a certain rational polyhedral cone $\mathcal{H}$. This cone $\mathcal{H}$ can be thought of as the Weil divisor analogue of the ample cone. In the case of general $\mathbb{H}$ we obtain, in a straight forward manner, that $\mathcal{H} \subseteq X(\overline{Z(\mathbb{H})})_0$.

3.1 The Cone $\mathcal{H}$

Throughout this section we use additive notation for any calculations with characters. Scalars are less cumbersome than exponents since we are are working with rational vector spaces.

Let M be a semisimple monoid with unit group G and semisimple part $G_0 \subseteq G$. By the results of § 2 we have a canonical quotient morphism

$$M_V \to M$$

and a canonical finite morphism

$$M_V \to M_{\mathbb{H}}$$

where $\mathbb{H} = \mathbb{H}(M)$. Since we shall be working in the appropriate rational cone, there is no harm in assuming that we actually have a morphism

$$M_{\mathbb{H}} \to M.$$

This amounts to replacing the δ in Proposition 3.1 below by $k\delta$ for some $k \in \mathbb{N}$. In any case, we can restrict characters, functions etc. on M to obtain like quantities on $M_{\mathbb{H}}$. For example, if $\delta \in X(M)$, we can write $\delta \in X(M_{\mathbb{H}})$.

Let $T_0 \subseteq G_0$ be a maximal torus and let $\overline{Z(\mathbb{H})} \subseteq M_{\mathbb{H}}$ be the Zariski closure of the connected center $Z(\mathbb{H})$ of the unit group of $M_{\mathbb{H}}$. Recall also the abelization $\pi : M_{\mathbb{H}} \to A_{\mathbb{H}}$ of $M_{\mathbb{H}}$. We can identify $X(A_{\mathbb{H}})$ as a submonoid of $X(\overline{Z(\mathbb{H})})$ by restriction of π. By Corollary 2.3 there is a structure map for $M_{\mathbb{H}}$,

$$\theta^* : X(T_0) \to X(\overline{Z(\mathbb{H})}).$$

Recall from the proof of Theorem 6.16 of [24], that

$$L(M_{\mathbb{H}}) = \{(\gamma, \lambda) \in X(\overline{Z}) \oplus X(T_0) \mid \gamma - \theta^*(\lambda) \in X(A_{\mathbb{H}})\}.$$

Proposition 3.1. *There is a unique $\delta \in X(\overline{Z(\mathbb{H})})$ such that*

$$L(M) = \{(k\delta, \lambda) \in X(\overline{Z(\mathbb{H})}) \oplus X(T_0) \mid k\delta - \theta^*(\lambda) \in X(A) \textit{ and } k \geq 0\}.$$

Proof. It is straightforward to check that

$$L(M) = L(M_{\mathbb{H}}) \cap \{(\gamma, \lambda) \in X(\overline{Z(\mathbb{H})}) \oplus X(T_0) \mid \gamma = k\delta \text{ for some } k \geq 0 \}.$$

δ is the generator of $X(M) \cong \mathbb{N}$, the character monoid of M. □

It is convenient to work over the positive rational numbers. Define

$$\mathcal{C} = X(T_0)_+ \otimes \mathbb{Q}^+,$$

the *rational Weyl chamber*.

Definition 3.2. If $\delta \in X(\overline{Z(\mathbb{H})})$ is as in Proposition 3.1 above we define

$$\mathcal{C}(\delta) = \{\lambda \in \mathcal{C} \mid \delta - \theta^*(\lambda) \in X(A)_0\}.$$

$\mathcal{C}(\delta)$ is called the *cross section polytope of M.*

$\mathcal{C}(\delta)$ is a fundamental domain for the action of the Weyl group W of G_0 on the polytope

$$\mathcal{P}(\delta) = \cup_{w \in W} w(\mathcal{C}(\delta)) \subseteq X(T_0) \otimes \mathbb{Q}.$$

As above let $\mathbb{H} = \mathbb{H}(M)$ be the arrangement of M and let $M_{\mathbb{H}}$ be the associated flat monoid with classifying map
$$\varphi : M_{\mathbb{H}} \to M.$$
Since φ is surjective $\varphi^* : K[M] \to K[M_{\mathbb{H}}]$ identifies $K[M]$ as a subalgebra of $K[M_{\mathbb{H}}]$. The action of $Z(\mathbb{H})$ on $M_{\mathbb{H}}$ determines a direct sum decomposition

$$K[M_{\mathbb{H}}] = \oplus_{\chi \in X(\overline{Z(\mathbb{H})})} K[M_{\mathbb{H}}]_{\chi}.$$

Let $Z \subseteq G$ be the connected center of the unit group of M. Since $dim(Z) = 1$, $K[M]$ is graded as follows.
$$K[M] = \oplus_{n \geq 0} K[M_{\mathbb{H}}]_{n\delta}$$
for some unique $\delta \in X(Z) \subseteq X(Z(\mathbb{H}))$. δ is the generator of $X(\overline{Z}) \cong \mathbb{N}$.

The major problem here is to characterize the subset of $X(\overline{Z(\mathbb{H})})$ consisting of all possible characters δ that can arise from a semisimple monoid N with $\mathbb{H}(N) = \mathbb{H}$. While this might seem indirect and removed from the underlying geometry, it is *the* question that motivated the current work. Proposition 3.3 and Theorem 3.4 below allow us to calculate this interesting cone for flat monoids of the form $M_{\mathbb{H}}$.

We denote the quadratic form on $\mathcal{C}$ by $(-,-)$.

Proposition 3.3. *For $\lambda \in \mathcal{C}$ we let $u_i^{\delta}(\lambda) = \nu_i(\delta - \theta^*(\lambda))$, where $\{\nu_i : X(Z(\mathbb{H})) \to \mathbb{Z} \mid i \in I\}$ is the set of essential valuations of $X(Z(\mathbb{H}))$ so that $X(\overline{Z(\mathbb{H})}) = \cap_i \{\chi \mid \nu_i(\chi) \geq 0\}$. Then*

1. *$\mathcal{C}(\delta) = \{\lambda \in \mathcal{C} \mid u_i^{\delta}(\lambda) \geq 0$ for all $i\}$.*
2. *For each i, there is a unique $r_i(\delta) \in \mathcal{C}$ such that $u_i^{\delta}(\lambda) = \epsilon_i(\delta)(r_i(\delta) - \lambda, r_i(\delta))$. Furthermore, $\epsilon_i(\delta) = \frac{\nu_i(\delta)}{(r_i(\delta), r_i(\delta))} > 0$.*

Proof. From Definition 3.2 we have that

$$\mathcal{C}(\delta) = \{\lambda \in \mathcal{C} \mid \delta - \theta^*(\lambda) \in X(A)_0\}.$$

But also $X(A)_0 = X(\overline{Z})_0 = \{\chi \in X(Z)_0 \mid \nu_i(\chi) \geq 0\}$. Thus,

$$\mathcal{C}(\delta) = \{\lambda \in \mathcal{C} \mid u_i^{\delta}(\lambda) \geq 0 \text{ for all } i\}.$$

This proves a).

Now let $H_i = \{\mu \in X(T_0) \mid \nu_i(\theta^*(\mu)) = 0\}$. Then $(H_i, l_i) \in \mathbb{H}$, where $l_i(\mu) = -\nu_i(\theta^*(\mu))$. So let
$$K(u_i^{\delta}) = \{\lambda \in \mathcal{C} \mid u_i^{\delta}(\lambda) = 0\}.$$
Then there exists a unique $r_i(\delta) \in \mathcal{C}$ such that

a) $\nu_i(\theta^*(r_i(\delta))) = \nu_i(\delta)$,

b) $(r_i(\delta), \mu) = 0$ for all $\mu \in H_i$.

We claim that $K(u_i^\delta) = r_i(\delta) + H_i$. Indeed, if $\mu \in H$, then we get $u_i^\delta(r_i(\delta) + \mu) = 0$ (using a) above and the fact that $\mu \in H_i$). Thus, $K(u_i^\delta) = r_i(\delta) + H_i$, since $K(u_i^\delta) \supseteq r_i(\delta) + H_i$, while they are both affine subspaces of the same dimension.

We claim also that there exists $\epsilon_i \in \mathbb{Q}^+$ such that $u_i^\delta(\lambda) = \epsilon_i(\delta)(r_i(\delta) - \lambda, r_i(\delta))$. Indeed, if $\lambda = r_i(\delta) + \mu \in K(u_i^\delta)$ then

$$(r_i(\delta) - \lambda, r_i(\delta)) = (-\mu, r_i(\delta)) = 0. \qquad (*)$$

Then we let

$$\epsilon_i(\delta) = \frac{\nu_i(\delta)}{(r_i(\delta), r_i(\delta))}.$$

Then for $\lambda \in r_i(\delta) + H_i$,

$$u_i^\delta(\lambda) = \epsilon_i(\delta)(r_i(\delta) - \lambda, r_i(\delta))$$

since by (*) the RHS is zero, and by definition the LHS is zero. But also,

$$u_i^\delta(0) = \nu_i(\delta) = \epsilon_i(\delta)(r_i(\delta) - 0, r_i(\delta)).$$

Thus, the formula is true for all λ. □

The following Theorem characterizes membership in the cone $\mathcal{H}$. It may not look like a "Theorem" but it yields an exact method for calculating $\mathcal{H}$ from $\mathbb{H}$.

Theorem 3.4. *Let $M = M_{\mathbb{H}}$. Then $\mathcal{H} = \{\delta \in X(\overline{Z(\mathbb{H})})_0 \mid \{r_i(\delta)\}$ satisfies the following condition$\}$.*

For each i there exists $x_i \in \mathcal{C}^0$, the interior of $\mathcal{C}$, such that

1. *$(r_j(\delta) - x_i, r_j(\delta)) > 0$ for all $j \neq i$.*
2. *$(r_i(\delta) - x_i, r_i(\delta)) = 0$.*

Proof. This is an exact reformulation, in terms of the r_i's, of the (nondegeneracy) condition that assures us that none of the H_i's is "lost" when restricting from $M_{\mathbb{H}}$ to M. It is equivalent to saying that, for each i, there is an $x_i \in (r_i(\delta) + H_i) \cap \mathcal{C}^0$, that is "below" the hyperplane $r_j(\delta) + H_j$ for each $j \neq i$. □

Corollary 3.5. *$\mathcal{H} + \mathcal{H} \subseteq \mathcal{H}$. Furthermore, for each i, $r_i(\delta + \gamma) = r_i(\delta) + r_i(\gamma)$.*

Proof. The basic idea here is this. If $\mathbf{x} = (x_i)$ works for δ, and $\mathbf{y} = (y_i)$ works for γ then $\mathbf{x} + \mathbf{y}$ works for $\delta + \gamma$.

Since both δ and γ come from the same collection $\mathbb{H}$ we obtain that, for each i,

$$r_i(\gamma) = \alpha_i r_i(\delta)$$

for some $\alpha_i \geq 0$. Now assume, as above, that $\mathbf{x} = (x_i)$ works for δ and $\mathbf{y} = (y_i)$ works for γ. By straight forward calculation we get that for each i,

$$(r_j(\delta) + r_j(\gamma) - (x_i + y_i), r_j(\delta) + r_j(\gamma)) > 0$$

if $i \neq j$, and

$$(r_i(\delta) + r_i(\gamma) - (x_i + y_i), r_i(\delta) + r_i(\gamma)) = 0.$$

Taking into account that, for each i, $r_i(\delta + \gamma) = r_i(\delta) + r_i(\gamma)$, the result follows. This can be checked using part 2 of Proposition 3.3. □

Definition 3.6. We refer to $\mathcal{H}$ as the *coterie* of $\mathbb{H}$ or of $M_{\mathbb{H}}$.

Remark 3.7. Let M be a reductive monoid. The **type map** $\lambda : \Lambda \to 2^S$ of M is a certain map from the set of $G \times G$-orbits of M to the set of subsets of the simple roots. This type map determines M, as an abstract monoid, to within a kind of central extension. This is equivalent to specifying the **colored face lattice** $\Lambda \subseteq 2^S \times 2^{\Lambda^1}$ of M considered as a spherical variety (see Proposition 5.20 of [24]). The type map of $M_{\mathbb{H}}$ is completely determined by the arrangement $\mathbb{H}$. Each type map of a semisimple monoid associated with $\mathbb{H}$ is a kind of "realization" of the type map of $M_{\mathbb{H}}$.

Remark 3.8. Let $\delta \in \mathcal{H}$ and let $X_\delta = (M \backslash \{0\})/K^*$. Then any ample Cartier divisor H on X_δ will allow one to recover X_δ from the graded algebra of global sections $\oplus_{n \geq 0} \Gamma(X_\delta, nH)$. This same graded algebra for an "ample" *Weil* divisor H on X_δ is naturally the cone on some other projective variety X_γ, a "morphed" version of X_δ. The resulting variety X_γ will agree with X_δ outside a closed subset of codimension two. $\mathcal{H}$ can be thought of as the set of all possible W-invariant, rational polytopes that can be constructed using $\mathbb{H}$ as the set of oriented hyperplanes associated with the facets.

The set of type maps partitions the cone $\mathcal{H} \subseteq Cl(X)$ into the disjoint union of potentially smaller ample Cartier cones $\mathcal{H}_\lambda$, each associated with some type map λ. Thus one may think of $\mathcal{H}$ as the cone of "ample Weil divisors" on $X_\delta = (M_\delta \backslash \{0\})/K^*$.

4 $\mathcal{H}$ for $Env(G_0)$

In this section we calculate explicitly the coterie $\mathcal{H}$ for $Env(G_0)$, where G_0 is a simple group. We obtain also some important information about the rational polyhedral cone $\overline{\mathcal{H}}$. However we first explain how $Env(G_0)$ is related to other important constructions.

Let L be a semisimple group of adjoint type, and suppose that $\sigma : L \to L$ is an involution (so that $\sigma \circ \sigma = id_L$) with $H = \{x \in L \mid \sigma(x) = x\}$. The **plongement magnifique** of L/H is the unique normal L-equivariant compactification X of L/H obtained by considering an irreducible representation $\rho : L \to Gl(V)$ of L with $\dim(V^H) = 1$ and with highest weight in general position. Then let $h \in V^H$ be nonzero and define

$$X = \overline{\rho(K)[h]} \subseteq \mathbb{P}(V),$$

the Zariski closure of the orbit of $[h]$. (See Section 2 of [3] for details.) We are here concerned only with the case where $L = G_0 \times G_0$, $\sigma(g, h) = (h, g)$ and G_0 is a simple group. This amounts to the situation where we consider two-sided compactifications of G_0, or **group embeddings**.

The relationship with semisimple monoids is as follows.

If G_0 is a simple group and M is a semisimple monoid with unit group $G_0 \times K^*$ then we say M is a **canonical monoid** if there exists a finite morphism

$$\rho : M \to End(V)$$

of algebraic monoids such that

a) ρ is irreducible considered as a representation of G_0.

b) the highest weight λ of ρ is of the form $\lambda = \sum_\alpha c_\alpha \lambda_\alpha$ where $\{\lambda_\alpha\}$ is the set of fundamental dominant weights, and each c_α is nonzero.

Canonical monoids are discussed in detail in [16]. There are many interesting characterizations of this class of monoids. It follows from the results of [21] or [3] that if G_0 is also of adjoint type then $(M\backslash\{0\})/K^*$ is isomorphic to the plongement magnifique.

But it is important for us to consider all simple groups, and not just adjoint groups, since geometric properties like smoothness are not involved in our calculations. Also our mission here is to calculate the rational cone $\mathcal{H}$ whose lattice points correspond to semisimple monoids with W-arrangement $\mathbb{H}_\Delta$. Here we define $\mathbb{H}_\Delta$, by setting

$$\Lambda^1\mathbb{H}_\Delta = \{(Span_{\mathbb{Z}}(\Delta\backslash\alpha), \nu_\alpha) \mid \alpha \in \Delta\},$$

where $\nu_\alpha : P \to \mathbb{Z}$ is defined by $\nu_\alpha(\beta) = -\delta_{\alpha\beta}\alpha$, and $\delta_{\alpha\beta}$ is the Kronecker delta. By Theorem 2.8 we obtain that

Theorem 4.1. *Let M be a semisimple monoid and let $\mathbb{H} = \mathbb{H}(M)$. The following are equivalent.*

1. $M_{\mathbb{H}} = Env(G_0)$.

2. $\mathbb{H}(M) = \mathbb{H}_\Delta$.

3. *There is a finite dominant morphism $M_V \to Env(G_0)$.*

In particular, if M is a canonical monoid, then $M_{\mathbb{H}} \cong Env(G_0)$. Indeed, one can check directly that, for a canonical monoid M, $\mathbb{H}(M) = \mathbb{H}_\Delta$. Thus $Env(G_0)$ is also the $\mathbb{Q}$-factorial hull of the plongement magnifique. As we shall see, there are many other semisimple monoids that have the same $\mathbb{Q}$-factorial hull as a canonical monoid. There are many a manifique wannabe.

We now proceed to the calculation.

Because of the special nature of canonical monoids and $Env(G_0)$ we are able to identify $\mathcal{H}$ as a subset of $X(T_0) \otimes \mathbb{Q}$ containing the interior of the (rational) Weyl chamber $\mathcal{C}^0$. This is convenient for our calculations, but it also depicts $\mathcal{H}$ as a kind of virtual Borel-Weil-Bott theorem for rank-one, locally free sheaves on $(M\backslash CD2)/K^*$, where $CD2 \subseteq M$ is the union of all $G \times G$-orbits of codimension two or more. The set of canonical monoids correspond to the points of $\mathcal{C}^0 \subseteq \mathcal{H}$. Indeed, we can get an inclusion

$$\mathcal{H} \subseteq P_0 \subseteq X(T_0) \otimes \mathbb{Q}$$

by observing that for each $\delta \in \mathcal{H}$ the cone $\mathcal{C}(\delta)$ is of the form

$$\mathcal{C}(\delta) = \mathcal{C} \cap (y - P_0)$$

for some unique $y = y(\delta) \in P_0$, where P_0 is the rational cone generated by the set of positive roots. This is so because the set of codimension-one colored faces of $\mathcal{C}(\delta)$ determines a collection of n hyperplanes ($n = dim(T_0)$) which intersect at exactly one point somewhere in $X(T_0) \otimes \mathbb{Q}$, but not necessarily in $\mathcal{C}(\delta)$.

Notice that $\mathcal{C} \subseteq P_0$. Thus, we obtain inclusions

$$\mathcal{C}^0 \subseteq \mathcal{H} \subseteq P_0 \subseteq X(T_0) \otimes \mathbb{Q}.$$

The second of these inclusions is defined, as above, by

$$\delta \rightsquigarrow y(\delta).$$

Associated with the simple group G_0 is its **Cartan matrix** C. The columns of $(C^{-1})^T$ are the coefficients that express the fundamental weights in terms of the simple roots. We refer to *Table 2* of [14] for the complete list of these coefficients.

Let $C_0(\Delta\backslash\{\alpha\})^0 = \{\sum_{\beta\neq\alpha} a_\beta\beta \in P_0 \mid a_\beta > 0$ for all $\beta \in \Delta\backslash\{\alpha\}\}$ be the interior of the rational cone $C_0(\Delta\backslash\{\alpha\})$.

Theorem 4.2. *Let X be the plongement magnifique for the simple group G_0. Let $P_0^0 \subseteq P_0$ be the interior of the cone P_0 (i.e. $x = \sum a_\alpha\alpha \in P_0$ such that $a_\alpha > 0$ for all α). Then in terms of the above identification $\mathcal{H} \subseteq P$,*

$$\mathcal{H} = \{x \in P^0 \mid \textit{for all } \alpha \in \Delta \textit{ there exists } r_\alpha(x) > 0 \textit{ such that } x - r_\alpha(x)\lambda_\alpha \in C_0(\Delta\backslash\{\alpha\})^0\}.$$

If we write $x = \sum_{\beta\in\Delta} a_\beta\beta \in P_0^0$ then the following are equivalent.

1. *$x \in \mathcal{H}$.*
2. *(a) $a_\beta > \frac{c_{\alpha,\beta}}{c_{\alpha,\alpha}} a_\alpha$ for all $\alpha \neq \beta$,*

   (b) $a_\alpha > 0$ for all $\alpha \in \Delta$.
3. *(a) $a_\beta > \frac{c_{\alpha,\beta}}{c_{\alpha,\alpha}} a_\alpha$ whenever $s_\alpha s_\beta \neq s_\beta s_\alpha$.*

   (b) $a_\alpha > 0$ for all $\alpha \in \Delta$.

Proof. By Lemma 7.2.2 of [24] there is a unique $e_\alpha \in \Lambda$ such that $\lambda(e_\alpha) = \Delta\backslash\alpha$. Furthermore, by Lemma 7.2.5 of [24] each e_α is in Λ^1 since

$$F_\alpha = (z_\alpha + H_\alpha) \cap \mathcal{P}(\delta)$$

is a codimension one face of $\mathcal{P}(\delta)$. Now each e_α has the property that

$$\mathbb{Q}^+\lambda_\alpha \cap F_\alpha = \{r_\alpha\lambda_\alpha\} \in F_\alpha^0$$

for some unique $r_\alpha \in \mathbb{Q}^+$, since $W_{\Delta\backslash\alpha}$ acts on F_α with $\{r_\alpha\lambda_\alpha\}$ as its unique fixed point.

But $\{r_\alpha\lambda_\alpha\} \in x - C_0(\Delta\backslash\{\alpha\})$ also, and thus,

$$\{r_\alpha\lambda_\alpha\} \in (x - C_0(\Delta\backslash\{\alpha\})) \cap F_\alpha^0 \subseteq x - C_0(\Delta\backslash\{\alpha\})^0.$$

Now let $x \in \mathcal{H} \subseteq P^0$. Then (as above) for each $\alpha \in \Delta$ there exists $r_\alpha(x) \in \mathbb{Q}$ such that

$$x - r_\alpha(x)\lambda_\alpha \in C_0(\Delta\backslash\{\alpha\})^0.$$

Write $x = \sum_{\beta\in\Delta} a_\beta\beta$ where $(C^T)^{-1} = (c_{\beta,\alpha})$ and C is the Cartan matrix. Thus

$$x - r_\alpha(x)\lambda_\alpha = \sum_\beta (a_\beta - r_\alpha(x)c_{\beta,\alpha})\beta$$

so that $a_\alpha - r_\alpha(x)c_{\alpha,\alpha} = 0$. Thus,

$$r_\alpha(x) = \frac{a_\alpha}{c_{\alpha,\alpha}}.$$

If $\beta \neq \alpha$ we obtain that

$$a_\beta - r_\alpha(x)c_{\beta,\alpha} > 0,$$

so that

$$a_\beta > \frac{c_{\beta,\alpha}}{c_{\alpha,\alpha}}a_\alpha$$

for all $\beta \neq \alpha$. We conclude that

$$\mathcal{H} = \{x = \sum a_\beta\beta \mid a_\beta > 0 \text{ and } a_\beta > \frac{c_{\beta,\alpha}}{c_{\alpha,\alpha}}a_\alpha \text{ for all } \beta \neq \alpha\}$$

It is sufficient to impose the condition "$a_\beta > \frac{c_{\beta,\alpha}}{c_{\alpha,\alpha}}a_\alpha$" in cases where $s_\alpha s_\beta \neq s_\beta s_\alpha$. This follows from the fact that if we have

$$\alpha - \beta - .... - \gamma$$

representing a subdiagram of the Dynkin diagram, then

$$c_{\alpha,\gamma} = \frac{c_{\alpha,\beta}}{c_{\beta,\beta}}c_{\beta,\gamma}.$$

This is easily checked by inspecting each matrix $(C^T)^{-1}$. See Table 2 of [14]. □

4.1 The Calculation of $\mathcal{H}$ for X in each Case

Using Theorem 4.2 we now calculate the cone $\mathcal{H}$ for the plongement manifique associated with each simple group G_0. In each case, we use a_j instead of a_{α_j}, with the numbering as dictated by the associated Dynkin diagram.

$\underline{A_n}$

$\mathcal{H} = \{x = \sum_j a_j\alpha_j \mid a_j$ as follows$\}$

1. $a_j > 0$.

2. $a_j > \frac{j}{j+1} a_{j+1}$ for $j < n$.

3. $a_j > \frac{n+1-j}{n+1-j+1} a_{j-1}$ for $j > 1$.

$\underline{B_n}$

$\mathcal{H} = \{x = \sum_j a_j \alpha_j \mid a_j \text{ as follows}\}$

1. $a_j > 0$.

2. $a_j > \frac{j}{j+1} a_{j+1}$ for $j < n$.

3. $a_j > a_{j-1}$ for $j > 1$.

$\underline{C_n}$

$\mathcal{H} = \{x = \sum_j a_j \alpha_j \mid a_j \text{ as follows}\}$

1. $a_j > 0$.

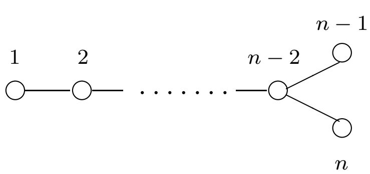
2. $a_j > a_{j-1}$ for $j < n$.

3. $a_j > \frac{j}{j+1} a_{j+1}$ for $j < n-1$.

4. $a_n > \frac{1}{2} a_{n-1}$.

5. $a_{n-1} > \frac{2(n-1)}{n} a_n$.

$\underline{D_n}$



$\mathcal{H} = \{x = \sum_j a_j \alpha_j \mid a_j \text{ as follows}\}$

1. $a_j > 0$.

2. $a_j > \frac{j}{j+1} a_{j+1}$ for $j < n-2$.

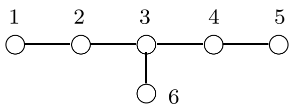
3. $a_{n-2} > \frac{2(n-2)}{n} a_{n-1}$

4. $a_{n-2} > \frac{2(n-2)}{n} a_n$

5. $a_n > \frac{1}{2}a_{n-2}$

6. $a_{n-1} > \frac{1}{2}a_{n-2}$

$\underline{E_6}$



$\mathcal{H} = \{x = \sum_j a_j\alpha_j \mid a_j \text{ as follows}\}$

1. $a_j > 0$.

2. $8a_1 > 4a_2 > 5a_1$.

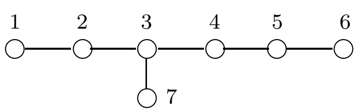
3. $5a_3 > 6a_2 > 4a_3$.

4. $5a_3 > 6a_4 > 4a_3$.

5. $4a_3 > 6a_6 > 3a_3$.

6. $8a_5 > 4a_4 > 5a_5$.

$\underline{E_7}$



$\mathcal{H} = \{x = \sum_j a_j\alpha_j \mid a_j \text{ as follows}\}$

1. $a_j > 0$.

2. $3a_1 > 4a_2 > 2a_1$.

3. $6a_2 > 4a_3 > 5a_2$.

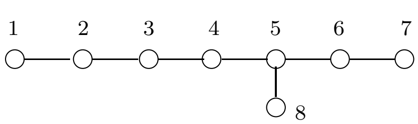
4. $10a_4 > 12a_3 > 9a_4$.

5. $9a_5 > 6a_4 > 8a_5$.

6. $7a_4 > 12a_7 > 6a_4$.

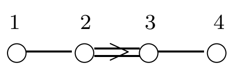
7. $4a_6 > 2a_5 > 3a_6$.

$\underline{E_8}$



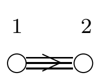
$\mathcal{H} = \{x = \sum_j a_j\alpha_j \mid a_j \text{ as follows}\}$

1. $a_j > 0$.
2. $4a_1 > 2a_2 > 3a_1$.
3. $9a_2 > 6a_3 > 8a_2$.
4. $16a_3 > 12a_4 > 15a_3$.
5. $25a_4 > 20a_5 > 24a_4$.
6. $16a_8 > 8a_5 > 15a_8$.
7. $21a_6 > 14a_5 > 20a_6$.
8. $8a_7 > 4a_6 > 7a_7$.

$\underline{F_4}$ 

$\mathcal{H} = \{x = \sum_j a_j\alpha_j \mid a_j \text{ as follows}\}$

1. $a_j > 0$.
2. $4a_1 > 2a_2 > 3a_1$.
3. $9a_2 > 12a_3 > 8a_2$.
4. $4a_3 > 6a_4 > 3a_3$.

$\underline{G_2}$ 

$\mathcal{H} = \{x = \sum_j a_j\alpha_j \mid a_j \text{ as follows}\}$

1. $a_j > 0$.
2. $4a_2 > 2a_1 > 3a_2$.

Notice that, for any rank-two example, $\mathcal{H} = \mathcal{C}$ the Weyl chamber.

4.2 The Structure of $\mathcal{H}$ and $\overline{\mathcal{H}}$

$\overline{\mathcal{H}}$ is a rational polyhedral cone. It is of interest to identify the face lattice $\mathcal{F}$ of $\overline{\mathcal{H}}$. It turns out that the faces of $\overline{\mathcal{H}}$ are indexed by orientations of the associated Dynkin diagram $\mathcal{D}$. To describe this correspondence let

$$\mathcal{E} = \{(\alpha, \beta) \in \Delta \times \Delta \mid s_\alpha s_\beta \neq s_\beta s_\alpha\}.$$

be the set of edges of $\mathcal{D}$ and let

$$\Gamma = \{\leftarrow, --, \rightarrow\} = \{l, n, r\}$$

be the realm of possible orientations (left, neutral, right) of each edge. Define

$$F = \{f \in Hom(\mathcal{E}, \Gamma) \mid f(\alpha, \beta) = r \text{ if and only if } f(\beta, \alpha) = l\}.$$

If $f \in F$ then, for each $(\alpha, \beta) \in \mathcal{E}$, $f(\alpha, \beta)$ represents either an arrow from α to β ($f(\alpha, \beta) = r$ and $f(\beta, \alpha) = l$), a broken line between α and β ($f(\alpha, \beta) = f(\beta, \alpha) = n$), or an arrow from β to α ($f(\alpha, \beta) = l$ and $f(\beta, \alpha) = r$). So we can think of these faces as diagrams like the following.

$$\alpha_1 \leftarrow \alpha_2 - -\alpha_3 \rightarrow \alpha_4 \leftarrow \alpha_5$$

We now define the ordering on F. If $f, g \in F$ we define $f \geq g$ if

a) $g(\alpha, \beta) = n$ implies that $f(\alpha, \beta) = n$,

b) $f(\alpha, \beta) = r$ implies that $g(\alpha, \beta) = r$ and

c) $f(\alpha, \beta) = l$ implies that $g(\alpha, \beta) = l$.

Thus $f \geq g$ if f can be obtained from g by replacing some of its arrows by broken lines. It is easy to check that $(F, \geq)$ is isomorphic to the face lattice of a cube. The vertices of F are those orientations $f \in F$ such that, for all $(\alpha, \beta) \in \mathcal{E}$, $f(\alpha, \beta) \neq n$.

Theorem 4.3. *Let G_0 be a simple group of rank n. There is a canonical one-to-one correspondence*

$$(F, \geq) \cong \mathcal{F}.$$

In particular, the vertex figure of $\overline{\mathcal{H}}$ is isomorphic to the face lattice of an $(n-1)$-cube.

Proof. In each case of rank n, $\mathcal{H}$ is defined by $n-1$ conditions of the form

$$ra_i > sa_j > ta_i.$$

Furthermore, the edges of $\mathcal{D}$ are exactly the pairs (i, j) that occur in each list of conditions. Now $\overline{\mathcal{H}}$ is defined by the $n-1$ conditions

$$ra_i \geq sa_j \geq ta_i. \qquad (*)$$

A face of $\overline{\mathcal{H}}$ is defined by replacing at most one "$\geq$" in each of these conditions by an "$=$". Replacing the left "$\geq$" by a "$=$" results in a $i \rightarrow j$. Replacing the right "$\geq$" by a "$=$" results in a $i \leftarrow j$. Finally, if no "$\geq$" in $(*)$ is replaced, then the result is a $i - -j$. This determines the canonical isomorphism $(F, \geq) \cong \mathcal{F}$ of partially ordered sets. $\square$

Example 4.4. In this example we calculate the extremal rays of $\overline{\mathcal{H}}$ in case G_0 is a simple group of type A_4. In each case we include the associated oriented Dynkin diagram, a representative of the corresponding extremal ray and the defining conditions for this extremal ray. It follows from Theorem 4.3 above that each of these extremal rays corresponds to an oriented diagram $f \in F$ such that $f(\alpha, \beta) \neq n$ for all $(\alpha, \beta) \in \mathcal{E}$.

1. $\alpha_1 \to \alpha_2 \to \alpha_3 \to \alpha_4$
   (a) $(1/4, 1/2, 3/4, 1)$
   (b) $2a_1 = a_2, 3a_2 = 2a_3, 4a_3 = 3a_4$.
2. $\alpha_1 \to \alpha_2 \to \alpha_3 \leftarrow \alpha_4$
   (a) $(2/3, 4/3, 2, 1)$
   (b) $2a_1 = a_2, 3a_2 = 2a_3, a_3 = 2a_4$.
3. $\alpha_1 \to \alpha_2 \leftarrow \alpha_3 \to \alpha_4$
   (a) $(9/16, 9/8, 3/4, 1)$
   (b) $2a_1 = a_2, 2a_2 = 3a_3, 4a_3 = 3a_4$.
4. $\alpha_1 \to \alpha_2 \leftarrow \alpha_3 \leftarrow \alpha_4$
   (a) $(3/2, 3, 2, 1)$
   (b) $2a_1 = a_2, 2a_2 = 3a_3, a_3 = 2a_4$.
5. $\alpha_1 \leftarrow \alpha_2 \to \alpha_3 \to \alpha_4$
   (a) $(2/3, 1/2, 3/4, 1)$
   (b) $3a_1 = 4a_2, 3a_2 = 2a_3, a_3 = 2a_4$.
6. $\alpha_1 \leftarrow \alpha_2 \to \alpha_3 \leftarrow \alpha_4$
   (a) $(16/9, 4/3, 2, 1)$
   (b) $3a_1 = 4a_2, 3a_2 = 2a_3, a_3 = 2a_4$.
7. $\alpha_1 \leftarrow \alpha_2 \leftarrow \alpha_3 \to \alpha_4$
   (a) $(3/2, 9/8, 3/4, 1)$
   (b) $3a_1 = 4a_2, 2a_2 = 3a_3, 4a_3 = 3a_4$.
8. $\alpha_1 \leftarrow \alpha_2 \leftarrow \alpha_3 \leftarrow \alpha_4$
   (a) $(4, 3, 2, 1)$
   (b) $3a_1 = 4a_2, 2a_2 = 3a_3, a_3 = 2a_4$.

References

[1] F. Berchtold, J. Hausen, *Cox rings and combinatorics*, Trans. A. M. S., to appear.

[2] M. Brion, *The behaviour at infinity of the Bruhat decomposition*, Comment. Math. Helv. 73(1998), 137-174.

[3] C. DeConcini, C. Procesi *Complete symmetric varieties*, Springer Lecture Notes 131(1983), 1-44.

[4] D. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geometry 4(1995), 17-50.

[5] E. Elizondo, K. Kurano, K. Watanabe, *The total coordinate ring of a normal projective variety*, J. of Alg. 276(2004), 625-637.

[6] B. Hassett, Y. Tschinkel, *Universal torsors and Cox rings*, *"Arithmetic of higher-dimensional algebraic varieties"*, 149-173, Progress in Math. 226, Birkhuser, (2003),

[7] J. Hausen, *Geometric Invariant Theory based on Weil Divisors*, Compositio Math. 140(2004), 1518-1536.

[8] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Mich. Math. J. 48(2000), 331-348.

[9] S. Kato, *A Borel-Weil-Bott type theorem for group completions*, Journal of Algebra 259(2003), 572-580.

[10] F. Knop, *Über Hilberts vierzehntes Problem für Varietäten mit Komplizierheit eins*, Math. Z. 213(1993), 33-35.

[11] D. Mumford, *Hilbert's fourteenth problem - The finite generation of subrings such as rings of invariants*, Proc. Symp. Pure Math. vol. 28, 1976, 431-444.

[12] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften 300, Springer-Verlag, Berlin, 1992.

[13] M. Nagata, *On the fourteenth problem of Hilbert*, Am. J. Math. 81(1959), 766-772.

[14] A. Onishchik, É. B. Vinberg, *"Lie Groups and Algebraic Groups"*, Springer Verlag, 1990.

[15] M. S. Putcha, *"Linear Algebraic Monoids"*, London Math. Soc. Lecture Notes, 1988.

[16] M. S. Putcha, L. E. Renner, *The canonical compactification of a finite group of Lie type*, Trans. Amer. Math. Soc. 337(1993), 305-319.

[17] M. S. Putcha and L. E. Renner *The orbit structure of 2-reducible algebraic monoids*, preprint, 2003.

[18] D. Rees, *On a problem of Zariski*, Ill. J. of Math. 2(1958), 145-149.

[19] L. E. Renner, *Admissible linearization of torus actions*, manuscript, 1984.

[20] L. E. Renner, *Classification of semisimple algebraic monoids*, Trans. Amer. Math. Soc. 292(1985), 193-223.

[21] L. E. Renner, *Classification of semisimple varieties*, J. of Alg. 122(1989), 275-287.

[22] L. E. Renner, *Modular Representatione of finite monoids of Lie type*, J. Pure and Apl. Alg. 138(1999), 279-296.

[23] L. E. Renner, *An explicit cell decomposition of the canonical compactification of an algebraic group*, Can. Math. Bull., 46(2003), 140-148.

[24] L. E. Renner, *"Linear Algebraic Monoids"*, Encyclopaedia of Mathematical Sciences, vol. 134, Subseries: Invariant theory and Transformation Groups vol. V, Springer Verlag, 2005.

[25] A. Rittatore, *Monoid algébrique et plongemont des groupes*, Thesis de Université Joseph Fourier, Grenoble, 1997.

[26] A. Rittatore, *Very flat reductive monoids*, Publ. Mat. Urug. 9(2001), 93-121.

[27] T. A. Springer, *Intersection cohomology of $B \times B$-orbit closures in group compactifications*, Journal of Algebra 258(2002), 71-111.

[28] L. Solomon, *An introduction to reductive monoids*, "Semigroups, formal languages and groups", J. Fountain, Ed., Kluwer Academic Publishers, 1995, 295-352.

[29] D. Timashev, *Equivariant compactifications of reductive groups*, Sbornik: Mathematics 194(2003), 589-616.

[30] É. B. Vinberg, *On reductive algebraic semigroups*, Amer. Math. Soc. Transl., Series 2, 169(1994), 145-182.

[31] O. Zariski, *Interprétation algebrico-géometriques du 14ième problème de Hilbert*, Bull. Sci. Math. 78(1954), 155-168.

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